SOLUTIONS

Math104/184 Final Exam (December 2010) [UBC]

1. (a) $\lim_{x\to 2}\dfrac{x^2+x-6}{3x^2-5x-2} = \lim_{x\to 2}\dfrac{(x+3)(x-2)}{(3x+1)(x-2)} = \lim_{x\to 2}\dfrac{x+3}{3x+1} = \dfrac{5}{7}$.

1. (b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[1+g(x)\cdot\dfrac{f(x)-1}{g(x)}\right] = \lim_{x\to\infty}\left[1+\dfrac{g(x)}{\frac{g(x)}{f(x)-1}}\right] = 1+\dfrac{\lim_{x\to\infty} g(x)}{\lim_{x\to\infty}\frac{g(x)}{f(x)-1}} = 1+\dfrac{6}{3} = 3.$

1. (c) Let A be the amount of money at time t. Then, since interest is compounded quarterly (i.e., four times a year), then $A = P(1+\frac{r}{n})^{nt}$, where $P=10{,}000$, $r=0.12$, and $n=4$. So

$$A = 10{,}000(1+\tfrac{0.12}{4})^{4t} = 10{,}000(1.03)^{4t}.$$

If $A = 12{,}000$, then

$$10{,}000(1.03)^{4t} = 12{,}000 \quad\Rightarrow\quad (1.03)^{4t} = 1.2 \quad\Rightarrow\quad 4t\ln 1.03 = \ln 1.2.$$

So $t = \dfrac{\ln 1.2}{4\ln 1.03} \approx 1.54$.

1. (d) The rate of growth is $R(t) = N'(t) = 60t - 3t^2$. This rate of growth will be decreasing if $R'(t)$ is negative. Since $R'(t) = 60-6t = 6(10-t)$, $R'(t)<0$ when $t>10$. So the rate of growth $N'(t)$ will be decreasing when $t>10$.

1. (e) $\lim_{x\to 3}\dfrac{x^2-3x}{f(3)-f(x)} = \lim_{x\to 3}\dfrac{x(x-3)}{-[f(x)-f(3)]} = \lim_{x\to 3}\dfrac{-x}{\frac{f(x)-f(3)}{x-3}} = -\dfrac{\lim_{x\to 3} x}{\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}} = -\dfrac{3}{f'(3)} = -\dfrac{3}{5}$

1. (f) If $y = \ln f(2x)$, then $\dfrac{dy}{dx} = \dfrac{1}{f(2x)}\cdot[f'(2x)\cdot 2] = \dfrac{2f'(2x)}{f(2x)}$.

At $x=1$, the slope of the graph of $y=\ln f(2x)$ is

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{2f'(2)}{f(2)} = \frac{2\cdot(-5)}{3} = -\frac{10}{3}.$$

1. (g) Since the tangent line to $y=f(x)$ at $x=3$ is $y=5x-7$, then the point $(3,8)$ is on both the tangent line and the curve, so $f(3)=8$ and $f'(3)=5$. Since we want to solve the equation $f(x)-x=0$, let $g(x)=f(x)-x$. Then $g'(x)=f'(x)-1$, so $g(3)=f(3)-3=8-3=5$ and $g'(3)=f'(3)-1=5-1=4$.

Thus $\quad x_1 = x_0 - \dfrac{g(x_0)}{g'(x_0)} = 3-\dfrac{g(3)}{g'(3)} = 3-\dfrac{5}{4} = \dfrac{7}{4} = 1.75$.

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1. (h) At $x=1$, $y=1\ln 1=0$, so the point of tangency is $(1,0)$.

Since $\dfrac{dy}{dx} = x\cdot\dfrac{1}{x}+\ln x\cdot 1 = 1+\ln x$, the slope of the tangent line to $y=x\ln x$ at $(1,0)$ is

$$m = \left.\frac{dy}{dx}\right|_{x=1} = 1+\ln 1 = 1+0 = 1.$$

The equation of the tangent line is therefore

$$y-0 = 1(x-1) \quad \text{or} \quad y = x-1.$$

1. (i) $$f''(x) = \frac{d}{dx}(f'(x)) = \frac{x\cdot\frac{2}{x}-2\ln x\cdot 1}{x^2} = \frac{2(1-\ln x)}{x^2}.$$

The function $f(x)$ is concave down when $f''(x)<0$. This happens when

$$\frac{2(1-\ln x)}{x^2}<0 \quad\Rightarrow\quad \ln x>1 \quad\Rightarrow\quad x>e^1=e.$$

So $f(x)$ is concave down on the interval $[e,\infty)$.

1. (j) $$x^2+2y^2=9 \quad\Rightarrow\quad \frac{d}{dt}(x^2+2y^2)=\frac{d}{dt}(9) \quad\Rightarrow\quad 2x\frac{dx}{dt}+4y\frac{dy}{dt}=0.$$

Plugging in $(x,y)=(1,2)$, $\dfrac{dx}{dt}=3$ gives

$$2\cdot 1\cdot 3+4\cdot 2\cdot\frac{dy}{dt}=0 \quad\Rightarrow\quad \frac{dy}{dt}=-\frac{6}{8}=-\frac{3}{4}=-0.75.$$

The *y*-coordinate is decreasing at a rate of 0.75 kilometres per minute.

1. (k) At a critical point, $f'(x)$ is either zero or undefined.

Now $f'(x)=\dfrac{2}{3}x^{-1/3}=\dfrac{2}{3\sqrt[3]{x}}$ can never be zero, but it *is* undefined when $x=0$. So $x=0$ is the only critical point. The endpoints are $x=-1$ and $x=8$.

Since $f(-1)=(-1)^{2/3}=(\sqrt[3]{-1})^2=1$, $f(0)=0^{2/3}=(\sqrt[3]{0})^2=0$ and $f(8)=8^{2/3}=(\sqrt[3]{8})^2=4$, the global maximum on the interval $[-1,8]$ is $f(8)=4$ and the global minimum is $f(0)=0$.

1. (l) $$f'(x) = a\cos 2x\cdot 2+(x\cdot(-\sin x)+\cos x\cdot 1) = 2a\cos 2x-x\sin x+\cos x.$$

If $x=\pi$ is a critical point of $f(x)$, then $f'(\pi)=0$, i.e.

$$f'(\pi)=2a\cos 2\pi-\pi\sin\pi+\cos\pi=0 \quad\Rightarrow\quad 2a\cdot 1-\pi\cdot 0+(-1)=0.$$

Therefore $2a-1=0$, so $a=\frac{1}{2}$.

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1. (m) $f(x)=e^{2x+1}$, $\qquad f(0)=e^1=e$,

$f'(x)=e^{2x+1}\cdot 2=2e^{2x+1}$, $\qquad f'(0)=2e^1=2e$,

$f''(x)=2e^{2x+1}\cdot 2=4e^{2x+1}$, $\qquad f''(0)=4e^1=4e$,

$f'''(x)=4e^{2x+1}\cdot 2=8e^{2x+1}$, $\qquad f'''(0)=8e^1=8e$.

The third degree Taylor polynomial is

$$T_3(x) \;=\; f(0)+f'(0)\,x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3 \;=\; e+2ex+\frac{4e}{2}x^2+\frac{8e}{6}x^3 \;=\; e+2ex+2ex^2+\tfrac{4}{3}ex^3.$$

Therefore $c_3=\tfrac{4}{3}e$.

1. (n) Let $f(x)=\ln x$, $a=1$. Then $f'(x)=\dfrac{1}{x}$. The linear approximation is given by

$$f(x) \;\approx\; f(a)+f'(a)(x-a) \;=\; f(1)+f'(1)(x-1) \;=\; \ln 1+\frac{1}{1}(x-1) \;=\; 0+(x-1),$$

so $\qquad \ln x \;\approx\; x-1$.

Therefore, $\ln 0.8\approx 0.8-1=-0.2$.

2. (a) $\dfrac{d}{dx}(x^3+xy^2+y^3)=\dfrac{d}{dx}(13) \quad\Rightarrow\quad 3x^2+(x\cdot 2yy'+y^2\cdot 1)+3y^2y'=0$

When $(x,y)=(1,2)$, this gives

$$3\cdot 1^2+(1\cdot 4y'+2^2)+3\cdot 2^2y'=0 \quad\Rightarrow\quad 3+4y'+4+12y'=0 \quad\Rightarrow\quad 16y'=-7.$$

The slope of the tangent line at the point $(1,2)$ is therefore $m=y'\big|_{(1,2)}=-\tfrac{7}{16}$.

The equation of the tangent line is $y-2=-\tfrac{7}{16}(x-1)$ or $y=-\tfrac{7}{16}x+\tfrac{39}{16}$.

2. (b) From part (a), the equation of the tangent line is

$$y-2=-\tfrac{7}{16}(x-1) \quad\Rightarrow\quad x-1=-\tfrac{16}{7}(y-2) \quad\Rightarrow\quad x=1-\tfrac{16}{7}(y-2).$$

If the y-coordinate is $\tfrac{31}{16}$, then the x-coordinate of the curve can be approximated by the x-coordinate of the tangent line, which is

$$x \;=\; 1-\tfrac{16}{7}(\tfrac{31}{16}-2) \;=\; 1-\tfrac{16}{7}(-\tfrac{1}{16}) \;=\; 1+\tfrac{1}{7} \;=\; \tfrac{8}{7}.$$

3. $\dfrac{d}{dp}(p^3+p)=\dfrac{d}{dp}(2q^3+q^2+10) \quad\Rightarrow\quad 3p^2+1 \;=\; 6q^2\dfrac{dq}{dp}+2q\dfrac{dq}{dp} \;=\; (6q^2+2q)\,\dfrac{dq}{dp}$

So $$\frac{dq}{dp} \;=\; \frac{3p^2+1}{6q^2+2q} \;=\; \frac{3p^2+1}{2q(3q+1)}.$$

The elasticity of supply when $p=3$ and $q=2$ is therefore

$$E \;=\; -\frac{p}{q}\frac{dq}{dp} \;=\; -\frac{p}{q}\cdot\frac{3p^2+1}{2q(3q+1)} \;=\; -\frac{3}{2}\cdot\frac{3\cdot 3^2+1}{2\cdot 2\cdot(6+1)} \;=\; -\frac{3}{2}\cdot\frac{28}{28} \;=\; -\frac{3}{2}.$$

4. The average cost is $A(q)=C(q)/q=3q^{1/3}+50+10{,}000q^{-1}$. Therefore

$$A'(q) \;=\; 3\cdot\tfrac{1}{3}q^{-2/3}+0+10{,}000\cdot(-q^{-2}) \;=\; \frac{1}{q^{2/3}}-\frac{10{,}000}{q^2},$$

At a critical point,

$$A'(q)=0 \quad\Rightarrow\quad \frac{1}{q^{2/3}}=\frac{10{,}000}{q^2} \quad\Rightarrow\quad \frac{q^2}{q^{2/3}}=10{,}000 \quad\Rightarrow\quad q^{4/3}=10^4$$

$$\Rightarrow\quad q^{1/3}=10 \quad\Rightarrow\quad q=10^3=1000.$$

The firm should produce 1000 kilograms in order to minimize the average cost.

5. The revenue is $R(q) \;=\; pq \;=\; (400-50q)\,q \;=\; 400q-50q^2$.

The profit, which is $P=R-C$, will be maximized at a critical point, where $\dfrac{dP}{dq}=0$, i.e.

$$\frac{dP}{dq} \;=\; \frac{dR}{dq}-\frac{dC}{dq} \;=\; (400-100q)-\frac{800}{q+5} \;=\; 100\left[(4-q)-\frac{8}{q+5}\right] \;=\; 0.$$

So $$4-q=\frac{8}{q+5} \quad\Rightarrow\quad (4-q)(q+5)=8 \quad\Rightarrow\quad 20-q-q^2=8$$

$$\Rightarrow\quad q^2+q-12=0 \quad\Rightarrow\quad (q-3)(q+4)=0.$$

So $q=3$ or $q=-4$. Since q must be positive, the solution is $q=3$. The manufacturer must sell 3000 jackets per month in order to maximize monthly profit.

6. The rate of increase of the price is

$$R(t) = \frac{dP}{dt} = 0 + 30\,[t \cdot (e^{-t/5} \cdot (-\tfrac{1}{5})) + e^{-t/5} \cdot 1] = 30e^{-t/5}\,(1 - \tfrac{1}{5}t) = 6e^{-t/5}\,(5 - t)\,.$$

In order to find the maximum value of *R*, it is necessary to check both the critical values of *R*, and the endpoints. Now

$$R'(t) = 6e^{-t/5} \cdot (0-1) + (5-t) \cdot (6e^{-t/5} \cdot (-\tfrac{1}{5})) = 6e^{-t/5}[-1 - \tfrac{1}{5}(5-t)]$$

$$= \tfrac{6}{5}e^{-t/5}[-5-(5-t)] = \tfrac{6}{5}e^{-t/5}(t-10)\,,$$

so at a critical point $\tfrac{6}{5}e^{-t/5}(t-10) = 0$. Therefore $t = 10$ is the only critical point.

When $t < 10$, $R'(t) = \tfrac{6}{5}e^{-t/5}(t-10) < 0$, and when $t > 10$, $R'(t) = \tfrac{6}{5}e^{-t/5}(t-10) > 0$.

So $R(t)$ is decreasing when $t < 10$ and increasing when $t > 10$. Therefore, $t = 10$ gives a local minimum. Therefore the maximum is at the endpoint $t = 0$. This is because $R(t) = 6e^{-t/5}(5-t)$ is positive when $t < 5$ and negative when $t > 5$. For $0 \le t < 5$, $R(t)$ is positive and decreasing, so its maximum value must occur at the left hand endpoint $t = 0$. The price of the stock is increasing most rapidly at $t = 0$.